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Computer Solutions of Wu's Equations for Compressible Flow Through Turbomachines

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Two computer programs, known as Matrix Through-Flow and Matrix Blade-To-Blade, for analyzing the meridional and blade-to-blade flow patterns are described. The numerical solutions are obtained by finite difference approximations to the governing Poisson-type differential equations for the stream function. Solutions for several turbomachines, giving flow patterns and velocity distributions, are included.

The flow through a modern gas turbine or compressor is an extremely complicated three-dimensional phenomenon. The flow has strong gradients in the three physical dimensions—axial, radial, and circumferential—as well as time and viscosity effects. The observation that the flow problem was not easily amenable to numerical solution led early investigators to search for a design system having ease of application. The computational difficulties were resolved by making approximations which permitted the use of two-dimensional techniques. These approximations were based on two flow models,

(1) Blade element flow
(2) Axially symmetric flow.

The blade element approach assumes that the flow in the blade-to-blade or circumferential plane can be described by considering the flow around blade profiles formed by the intersection of a cylindrical flow surface and the blading.

Axial symmetry assumes that an average value can be utilized to represent the state of the fluid in the blade-to-blade plane.

On the basis of these two flow models, several investigators developed analysis and design methods for the axial-flow compressor and turbine. In the case of the compressor, one of the earliest design methods appeared in Howell's classic papers in 1945 (refs. 1 and 2). Using the blade element flow model, Howell correlated experimental linear cascade data to establish a limit that has to be placed on the allowable deflection in any one blade row and determined empirical rules for the deviation and flow loss. In estimating the overall performance of the compressor, the flow is analyzed along a "mean" or "reference" diameter and the gas state is estimated at planes between adjacent blade rows, making use of the axially symmetric flow model. Similar methods were developed for the axial-flow turbine and, of these, the method of Ainley and Mathieson (ref. 3) is one of the best known. These relatively simple, albeit one-dimensional methods for analyzing the overall properties of the flow field, developed when the digital computer was in its infancy and the development of methods suitable for hand desk machines was one of the prime goals, are still, in principle, used widely throughout the aircraft industry and are likely to remain in use for some time.

More recently, with the advent of the large, high-speed digital computer, techniques (refs. 4 and 5) have been developed for analyzing the subsonic fluid motion in the meridional or hub-to-tip plane of axial-flow machines at stations other than the mean diameter (which was used in the early days) both inside the blade rows and in the duct regions. Similar methods have been developed for centrifugal and mixed-flow impellers by Hodskinson (ref. 6) and Wood, et al. (ref. 7). In parallel, several investigators (refs. 8, 9, and 10) have been working on the problem of generating a computer solution for the subsonic blade-to-blade flow with allowances for radial acceleration imposed by the curvature of the streamlines in the meridional plane and for the effects of Coriolis forces.

The purpose of this paper is to present an outline of two advanced computer solutions that have been developed at the National Gas Turbine Establishment (NGTE) for the meridional and blade-to-blade flow patterns. Solutions for several turbomachines, giving flow patterns and velocity distributions, are included.

MATHEMATICAL ANALYSIS

The mathematical analysis is based on the earlier work of Wu (ref. 11) who developed a general theory for the three-dimensional, inviscid, steady flow through an arbitrary turbomachine. The equations of motion are satisfied on two intersecting families of stream surfaces known as the first kind, $S1$ (blade-to-blade), and the second kind, $S2$ (meridional), the complete flow solution being obtained by an iterative process between the flows in the two stream surfaces.

Stream Function Equation for $S1$ Surface

In the real blade-to-blade flow, the $S1$ stream surface would be twisted. To permit computations of the potential flow in the blade-to-blade plane of a stationary or rotating blade row, Smith (refs. 10 and 12) assumed the stream surface was a surface of revolution.

The shape of the $S1$ surface is obtained by rotating a streamline in the meridional plane (fig. 1) about the axis of rotation.[1] In order to analyze the flow through any type of turbomachine, it is convenient to rotate the r,z axes through an angle θ. Using x and ϕ as the two independent variables, the continuity equation and the equations of motion can be manipulated to arrive at a Poisson-type differential equation for the stream function.

$$\frac{\bar{\partial}^2\psi}{\partial x^2}+\frac{1}{r^2}\frac{\bar{\partial}^2\psi}{\partial\phi^2}=F(x,\phi) \tag{1}$$

where the stream function ψ is defined by

$$\left.\begin{aligned} \frac{1}{r}\frac{\bar{\partial}\psi}{\partial\phi}&=b\rho W_x \\ \frac{\bar{\partial}\psi}{\partial x}&=-b\rho W_\phi \end{aligned}\right\} \tag{2}$$

and the velocity components W_x and W_y are related by

$$W_y=-W_x\tan\lambda \tag{3}$$

Equation (3) is the geometrical condition that the flow follows the stream surface. The derivatives in equations (1) and (2) are those which Wu

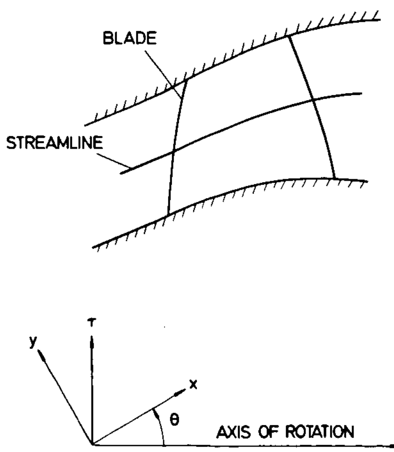


FIGURE 1.—*Meridional plane.*

[1] For two-dimensional cascade flow the stream surface is a cylinder.

refers to as special derivatives taken on the stream surface, and the integrating factor b is proportional to the local thickness of a thin stream sheet whose mean stream surface is the $S1$ surface considered.

Stream Function Equation for $S2$ Surface

To analyze the flow in the meridional plane of an arbitrary turbomachine, Marsh (ref. 4) developed a matrix through-flow method. Inside the blade rows, the flow is analyzed in an $S2$ stream surface and for the duct regions between adjacent blade rows, the flow is assumed to be axially symmetric.

As in the case of the $S1$ surface, the r,z axes (fig. 1) are rotated through an angle θ and x,y are the two independent variables. In a manner similar to the $S1$ solution, an equation for the stream function can be derived.

$$\frac{\bar{\partial}^2\psi}{\partial x^2}+\frac{\bar{\partial}^2\psi}{\partial y^2}=f(x,y) \tag{4}$$

where the stream function satisfies

$$\left.\begin{aligned} \frac{\bar{\partial}\psi}{\partial x}&=-rB\rho W_y \\ \frac{\bar{\partial}\psi}{\partial y}&=rB\rho W_x \end{aligned}\right\} \tag{5}$$

The integrating factor B in equation (5) is proportional to the local angular thickness of the $S2$ stream surface and in the through-flow analysis it is assumed to be proportional to the width of the blade passage. In formulating the stream function equation—equation (4)—the viscosity terms were omitted in the equations of motion but the entropy terms were included, and Marsh introduced the effects of irreversibility into the flow calculation by defining a local polytropic efficiency for expansion and compression.

For the flow to follow the stream surface, within the blade rows, the three components of velocity are related by

$$W_\phi=-W_y\tan\lambda-W_x\tan\mu \tag{6}$$

In the duct regions there is no change of angular momentum along a streamline and the circumferential velocity satisfies the relationship

$$rV_\phi=\text{constant} \tag{7}$$

NUMERICAL SOLUTION

The equations for the stream function—equations (1) and (4)—are nonlinear, but they can be solved iteratively using finite difference techniques.

Finite Difference Approximations

In conventional finite difference analysis the domain is covered with a square or rectangular grid and a five-point star is used since this leads to a simple approximation for the Laplacian operator. However, for analyzing the flow in the $S1$ and $S2$ stream surfaces such a simple grid is not accurate enough, owing to the irregular boundaries of the flow domain giving rise to boundary finite difference stars with short limbs and consequently a large truncation error. A good example of this, in fluid mechanics, is the recent blade-to-blade method developed by Katsanis (refs. 8 and 9) in which the flow domain is covered with a square grid. It is clear that the truncation error is significant since the boundary condition of zero velocity normal to the blade surfaces is not satisfied.

In the NGTE methods, use is made of the powerful software of present-day digital computers by adopting an asymmetric finite difference grid. The grid (fig. 2) consists of straight lines normal to the x direction, each line having the same number of equally spaced grid points. In the case of the $S1$ surface, the blade suction and pressure surfaces form curved grid lines, and for the $S2$ surface, the inner and outer annulus walls form curved grid lines so that there are no additional difficulties for grid points close to the boundaries. The spacing of the straight lines need not be uniform and where necessary can be varied locally (in the blade leading and trailing edges, for instance) in order to obtain a detailed picture of the flow.

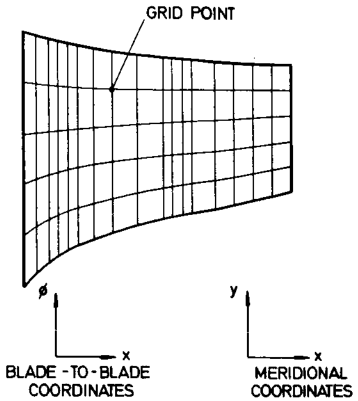


FIGURE 2.—*Asymmetric grid.*

To formulate a finite difference approximation for the Laplacian operator having an error of k^2, where k is the local grid spacing, equations (1) and (4) were modified by adding the term $E(\bar{\partial}\psi/\partial x)$ to both sides. Thus, for the $S1$ surface the stream function equation becomes

$$\frac{\bar{\partial}^2\psi}{\partial x^2}+\frac{1}{r^2}\frac{\bar{\partial}^2\psi}{\partial\phi^2}+E\frac{\bar{\partial}\psi}{\partial x}=F(x,\phi)+E\frac{\bar{\partial}\psi}{\partial x}$$

or

$$\nabla^2\psi+E\frac{\bar{\partial}\psi}{\partial x}=q(x,\phi) \tag{8}$$

where E is a function of the grid spacing in the x direction and is zero for uniform spacing. The operator $\nabla^2\psi+E(\bar{\partial}\psi/\partial x)$ is approximated by a ten-point star for the interdependence of the function values at neighboring grid points. To maintain an overall accuracy of order k^2, the derivative $\bar{\partial}\psi/\partial x$ is also approximated by the use of a ten-point star.

Boundary Conditions

Considering first the $S2$ surface, the boundary conditions are relatively simple. At inlet to the turbomachine, the flow conditions are known; therefore, the stream function distribution is defined for the first straight line of the grid. The inner and outer annulus walls form limiting streamlines, so that for grid points on the walls the stream function is known. For the far downstream boundary, it is assumed that the shape of the exit duct is such that the stream function distribution is the same on the last two straight lines of the grid.

The blade-to-blade problem—$S1$ surface—poses quite complex boundary conditions. Far upstream of the blade row the gas state and flow angle are known and it is assumed that the flow is uniform. The gradient of stream function is defined, therefore, for the first straight line of the grid. Thus, from equation (2)

$$\left(\frac{\bar{\partial}\psi}{\partial x}\right)_u=-\frac{Q}{r_u}\,\Delta\phi\tan\alpha_u$$

where $\Delta\phi=2\pi/N$ and N is the number of blades in the row. For the blade region the suction and pressure surfaces form, by definition, limiting streamlines so that for grid points on the blade surfaces the stream function is known. Upstream and downstream of the blade the locations of the streamlines are not known until the problem is solved. For these regions, the boundary condition is that there is a circumferential periodicity of the flow. The final condition is that for the far downstream boundary. In a real blade-to-blade flow the circulation, and consequently the outlet flow angle, is largely controlled by viscosity. In a potential flow model a criterion has to be adopted for fixing the circulation. In the method developed at NGTE, it is assumed that the flow is uniform and the flow angle is known far downstream of the blade row. These conditions fix the gradient of stream function on the last straight line of the grid which, from equation (2), is

$$\left(\frac{\bar{\partial}\psi}{\partial x}\right)_d = \frac{Q}{r_d}\,\Delta\phi \tan \alpha_d$$

Solution of Banded Equations and Convergence

By making use of the finite difference approximations and the boundary conditions, the modified stream function equations—equation (8) for the $S1$ surface—can be written in matrix form:

$$[M]\cdot[\psi]=[q] \tag{9}$$

where $[\psi]$ and $[q]$ are column vectors formed by ψ and q at each grid point and $[M]$ is a band matrix of the influence coefficients of the finite difference approximations. The method of solving equation (9) for the stream function is to solve for a given vector $[q]$, to correct $[q]$ using the new flow pattern, and then to repeat the cycle of calculation until the solution has converged to a specified tolerance. Since the matrix $[M]$ is "banded," only the band of nonzero elements is formed and stored in the computer and a very efficient direct method (ref. 13) is used to solve equation (9) for a given vector $[q]$. This method is better than the alternative indirect or relaxation method, as used by Katsanis, for the simple reason that it is very stable numerically.

Numerical stability can be a major problem with any iterative method. In the matrix through-flow and blade-to-blade methods, the iterative process has been made stable by introducing a relaxation factor R; thus,

$$\psi_p=\psi_{p-1}+R(\psi-\psi_{p-1}) \tag{10}$$

where

ψ calculated value for the pth iteration
ψ_p value taken for the pth iteration
ψ_{p-1} value taken for the $(p-1)$th iteration.

Additional stability was obtained in the through-flow method by limiting the percentage change in ψ between successive iterations, a restriction which is automatically removed as the solution converges. For the blade-to-blade method, the stability was further improved for compressible flow by adopting a "marching" process of increasing the inlet Mach number gradually to the required value.

When the stream function is known, it is possible to calculate the products ρW_x, ρW_ϕ and ρW_y. To calculate the density and hence the velocity components a tabular method, as developed by Wu, is used in the through-flow method. For the blade-to-blade problem, an alternative method, suggested by Gelder (ref. 14), is used. In this method, the calculation of density is allowed to lag the stream function calculation by one iteration. This has the effect of improving stability and for compressible flow, the relaxation factor R—equation (10)—is a function of the maximum Mach number.

BLADE-TO-BLADE FLOW PATTERNS

Eight examples are given to illustrate the use of the blade-to-blade computer program.

(1) Impulse turbine cascade
(2) Seventy-degree camber blade
(3) Axial turbine rotor tip section
(4) Axial turbine rotor root section
(5) Axial turbine stator blade
(6) Turbine stator cascade
(7) Three-dimensional flow past turbine stator blade
(8) Radial cascade diffuser

Impulse Turbine Cascade

The first example is the incompressible flow past a 112-degree camber blade in cascade. The blade profile (fig. 3) is an impulse-type turbine blade having a pitch/chord ratio of 0.59 and 101 degrees flow deflection.

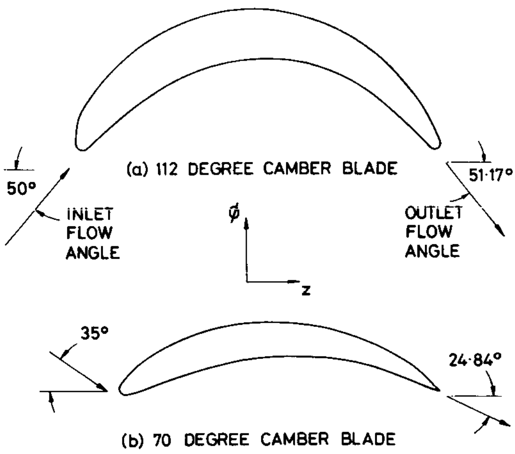


FIGURE 3.—*Blade profile-exact solutions.*

The velocity ratios are plotted in figure 4. (Velocity ratio is defined as the ratio of local surface velocity to far downstream velocity.) Also shown is an exact solution obtained by Gostelow (ref. 15). The matrix solution is in very good agreement with the exact solution.

Seventy-Degree Camber Blade

This blade profile (fig. 3) has a pitch/chord ratio of 0.9 and 70 degrees of camber. The two-dimensional, incompressible velocity distribution for −70 degrees of incidence is compared with an exact Gostelow solution in figure 5. In general, the matrix solution is in excellent agreement with the

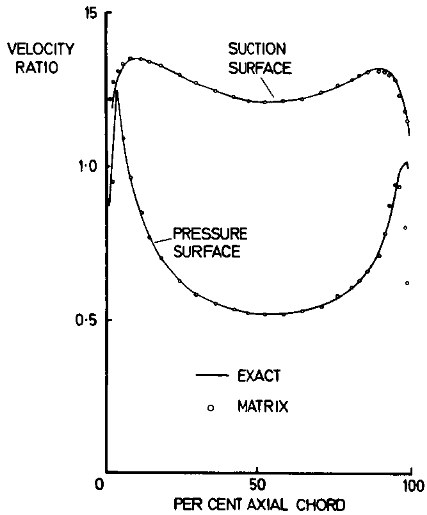


FIGURE 4.—*Velocity distribution for impulse-type blade.*

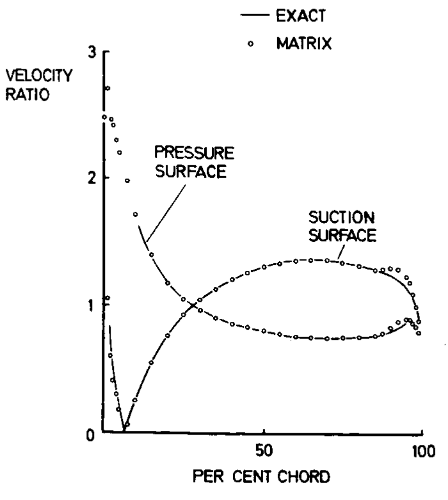


FIGURE 5.—*Velocity distribution for 70-degree camber blade.*

exact solution. It is noticeable that the main discrepancies are in the region of the blade trailing edge on the suction surface. For this region, the exact profile coordinates are a long way apart and it is probable that errors in interpolating the coordinates for the matrix solution have caused the discrepancies. There seems no reasonable doubt that complete agreement would have been obtained if the exact airfoil shape had been more fully defined. This example shows that there is no problem in analyzing high-incidence flows. The streamline pattern, calculated by the matrix method, is shown in figure 6. It may be seen that the leading edge stagnation point is well round on the suction surface.

Axial Turbine Rotor Tip Section

This example of two-dimensional, incompressible flow past a rotor tip section is given to illustrate the type of detailed flow pattern that can be calculated. The blade section is typical of a high pressure ratio turbine stage and is formed by a parabolic camber line and an analytical thickness distribution (ref. 16). Initially, the profile was designed so that the blade inlet angle was equal to the gas inlet angle of 18 degrees, a condition often referred to as zero geometric incidence. Figure 7 shows the blade profile. The surface velocity distribution around the blade leading edge is plotted in figure 8. It may be seen that it has the undesirable characteristic of a high peak on the suction surface. Such effects have been found by Hall (ref. 17). This is due to the large induced incidence which can be seen from the streamline pattern in figure 9a. The high suction peak was reduced by effectively drooping the nose of the blade (fig. 7) by 10 degrees so that the profile was operating at −10 degrees geometric incidence. The resulting velocity distribution is shown in figure 8 and, from the streamline pattern (fig. 9b), it may be seen that the induced incidence was considerably reduced. These results serve to show that computer methods can be very powerful in analyzing detailed aspects of the flow which would probably be very difficult to find experimentally.

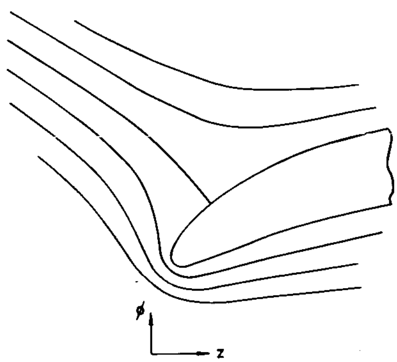


FIGURE 6.—*Streamline pattern for leading edge of 70-degree camber blade.*

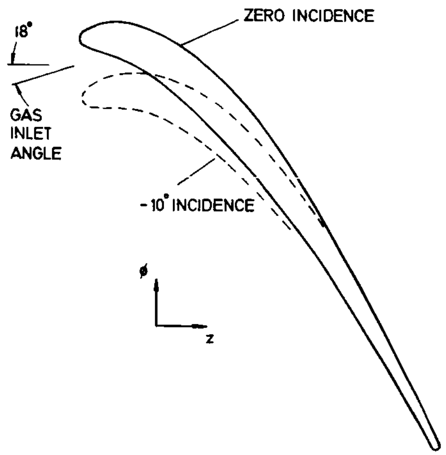
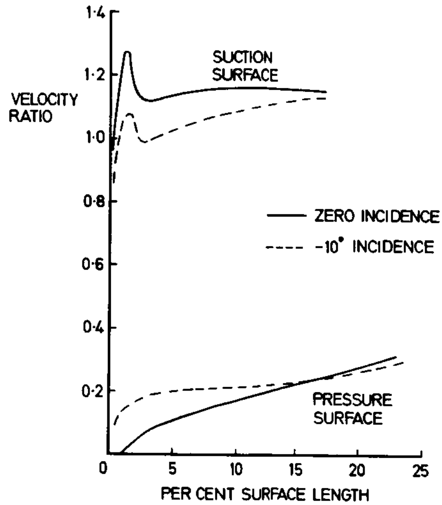


FIGURE 7.—*Turbine rotor tip sections.*

FIGURE 8.—*Velocity distributions for turbine rotor tip sections.*



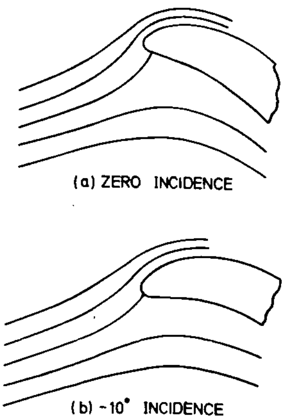
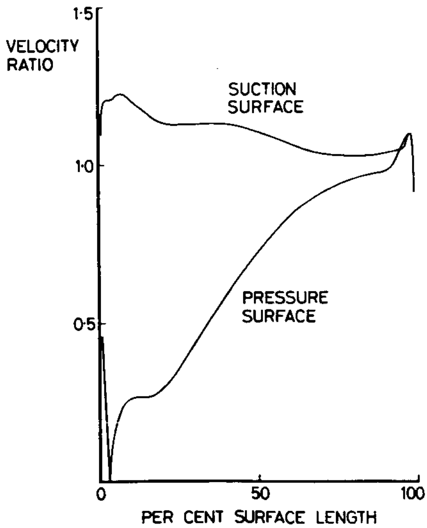


FIGURE 9.—*Streamline patterns for leading edge of turbine rotor tip sections.*

Axial Turbine Rotor Root Section

This blade section is typical of a high pressure ratio turbine stage. The basis for the design is the same as that of the previous example. The gas inlet angle was 48.9 degrees and the blade geometry at inlet was chosen so that the geometric incidence was zero. At outlet, the blade passage was adjusted to satisfy the gas outlet angle of −63.9 degrees by the empirical rule of Ainley and Mathieson (ref. 3). The blade surface velocity for two-dimensional, incompressible flow (fig. 10) shows that a detailed solution can be obtained in the region of the leading edge stagnation point. A particularly interesting feature of this blade section is that, according to the Ainley and Mathieson rule, the deviation[2] is 2.87 degrees negative. The velocity distribution for the trailing edge region is shown, enlarged, in figure 11 for an outlet flow angle of −64.15 degrees—a difference of only 0.25 degrees from the Ainley and Mathieson value. It is seen that on both the suction and pressure surfaces there is a rapid rise in velocity as the flow passes around the trailing edge. At the blade cutoff points, the velocities are equal, a criterion often used for fixing the outlet flow angle (ref. 18). Also, if the two surface velocity distributions are extrapolated then the loading at the blade trailing edge is zero, thus satisfying Preston's theorem (ref. 19) that equal and opposite vorticity should be shed from

FIGURE 10.—*Velocity distribution for turbine rotor root section.*



[2] Deviation is the difference between the fluid and blade outlet angles.

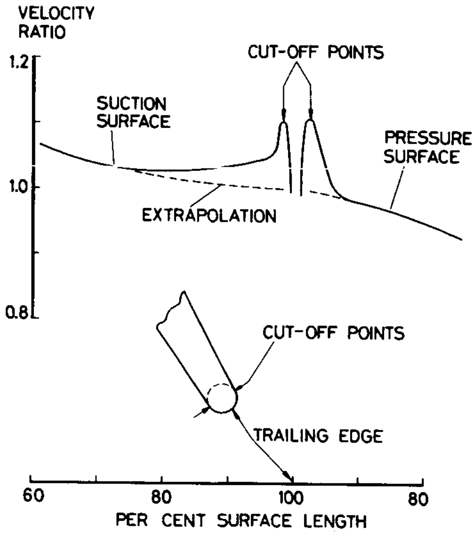


FIGURE 11.—*Trailing edge velocity distribution for turbine rotor root section.*

the two blade surfaces. The streamline pattern is shown in figure 12. It is seen that the flow leaves the trailing edge smoothly. This example shows that, applying existing velocity distribution criteria, the potential flow model gives an outlet flow angle in good agreement with well-established empirical rules, although it is perhaps surprising to find that the deviation is negative.

Axial Turbine Stator Blade

This example of two-dimensional, compressible flow is for the mean diameter section of a stator for a NASA turbine (ref. 20) operating at the design mass flow. The theoretical and experimental distributions of blade surface Mach number are compared in figure 13. In general, the computed Mach numbers agree well with experimental data. As mentioned earlier, Katsanis has developed a similar blade-to-blade method. In his recent paper (ref. 21) mention was made of an attempt to analyze the flow past this blade. He found that it was not possible to obtain an exact solution[3] and he had to resort to an approximate solution.

Turbine Stator Cascade

This turbine cascade was fitted with blades having the same profile as the mean diameter section of the second-stage stator blades of the turbine

[3] The term "exact" has been used as meaning a numerical solution from the computer program.

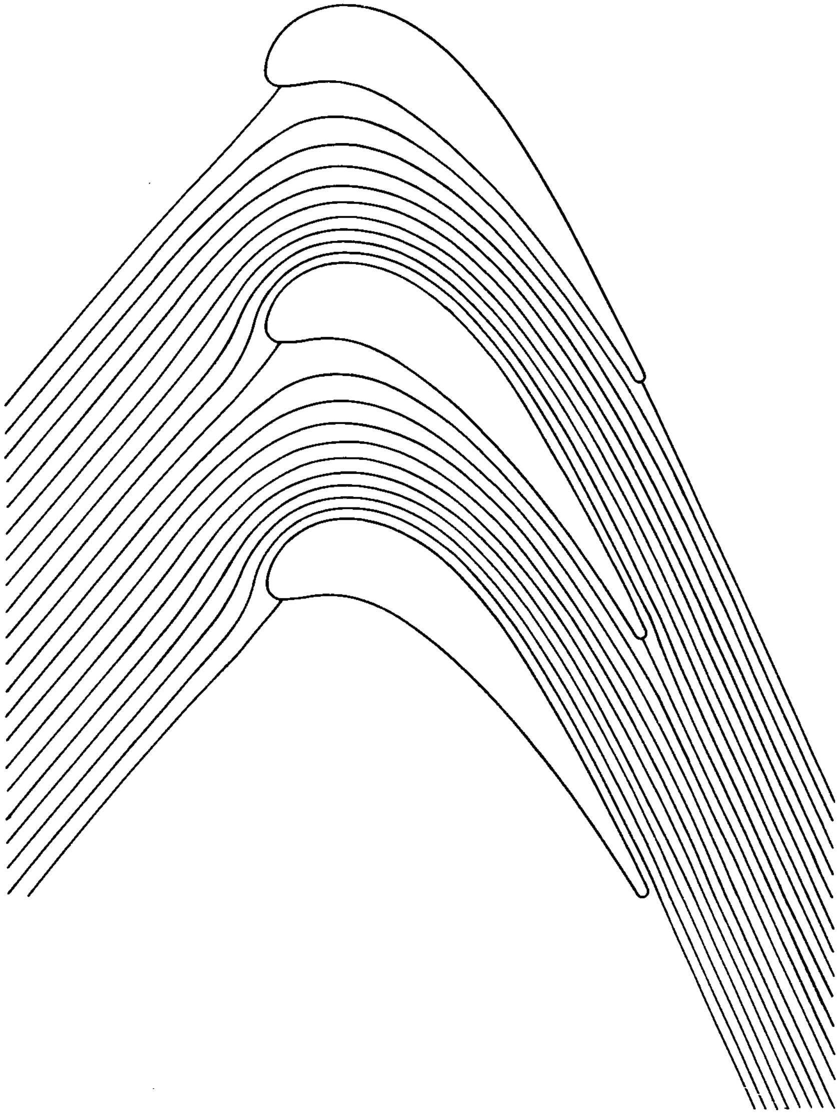


FIGURE 12.—*Computer output—streamline pattern for turbine rotor root section.*

described in reference 22. Two compressible flow solutions for the blade surface Mach number distribution are compared with experimental data in figure 14. The computed Mach numbers agree well with experimental data for this example.

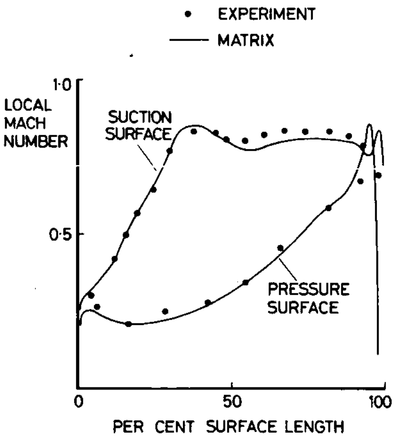


FIGURE 13.—*Mach number distribution for NASA axial-flow turbine.*

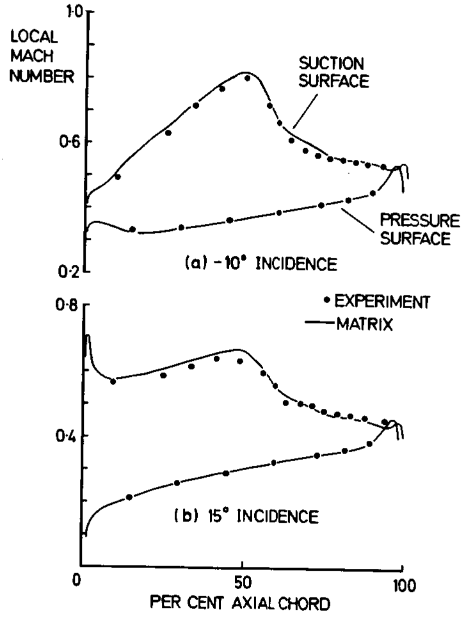


FIGURE 14.—*Turbine cascade.*

Three-Dimensional Flow Past Turbine Stator Blade

The flow through the two-stage turbine mentioned in the previous example has been analyzed using both the blade-to-blade and through-flow programs. The results of the through-flow analysis are presented in a later section of this paper.

Two matrix blade-to-blade solutions for the flow past the second-stage stator blades were computed. The first solution was for two-dimensional flow (i.e., cylindrical stream surface of constant thickness) and the outlet flow angle was calculated from the Ainley and Mathieson empirical rule. The second or quasi-three-dimensional solution is a refinement in that the stream surface thickness was varied. The variation of thickness was determined from a solution for a meridional flow pattern using the through-flow program and the outlet flow angle was determined by applying the condition of zero trailing edge loading.

A comparison of observed blade surface Mach numbers with the theoretical calculations, for the mean diameter section, is shown in figure 15. The most striking point here is that when some of the interactions between the meridional and blade-to-blade flow patterns are introduced the quasi-three-dimensional solution is in good agreement with experimental data. As mentioned earlier, this mean diameter section has been tested in cascade. The cascade Mach number distribution shown in figure 14a corresponds to the turbine flow conditions given in figure 15. By comparing the cascade and turbine results, it may be seen that the three-dimensional flow effects are significant on the peak surface Mach number.

Radial Cascade Diffuser

To illustrate the types of turbomachines to which the matrix blade-to-blade method can be applied, the last example is a radial cascade diffuser. The initial calculations were made for incompressible flow with the cascade operating at zero geometric incidence and the outlet flow angle equal to the blade outlet angle (i.e., zero deviation). The theoretical velocity distribution is shown in figure 16. The peak near the trailing edge is due to the potential flow model picking up the rapid change in blade surface curvature in this region. In real flow, such peak velocities would be removed by the presence of boundary layers. By extrapolating the suction

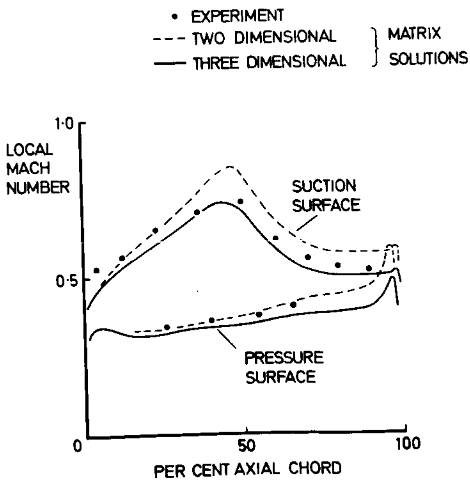


FIGURE 15.—*Blade surface Mach numbers for two-stage turbine stator blade.*

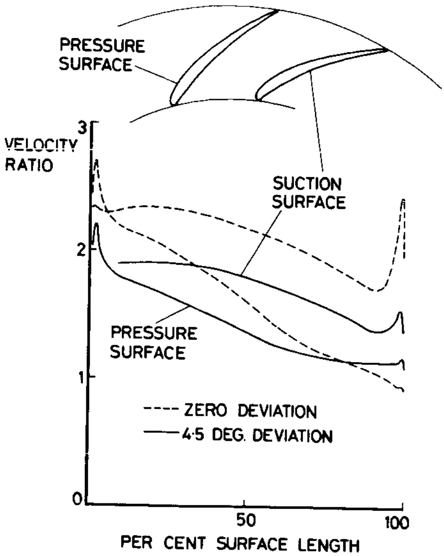


FIGURE 16.—*Velocity distribution for a radial cascade diffuser.*

surface velocity distribution from 90 percent of the way along the surface (i.e., just upstream of the rapid acceleration in the trailing edge region), it is seen that the condition of zero trailing edge loading is not satisfied. By increasing the outlet flow angle such that the deviation is 4.5 degrees, it is seen that the loading at the trailing edge satisfies Preston's theorem.

MERIDIONAL FLOW PATTERNS

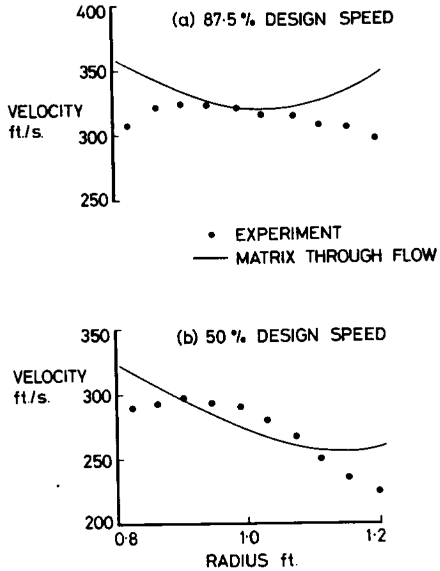
In this section, four examples are given of meridional flow patterns obtained from the matrix through-flow program.

(1) Two-stage axial-flow turbine
(2) Single-stage axial-flow turbine
(3) Low pressure ratio centrifugal compressor
(4) High pressure ratio centrifugal compressor

Two-Stage Axial Flow Turbine

This turbine (ref. 22) is the one referred to in the previous section. In applying the matrix through-flow program, the effects of irreversibility were taken into account by assuming that the local polytropic efficiencies were constant throughout the flow field. From the comparison of the experimental and predicted profiles of axial velocity at the turbine exit shown in figure 17, it is seen that the through-flow theory gives a fair estimate of the axial velocities. Recent work by Gregory-Smith (ref. 23)

FIGURE 17.—*Axial velocity profiles far downstream of two-stage turbine.*



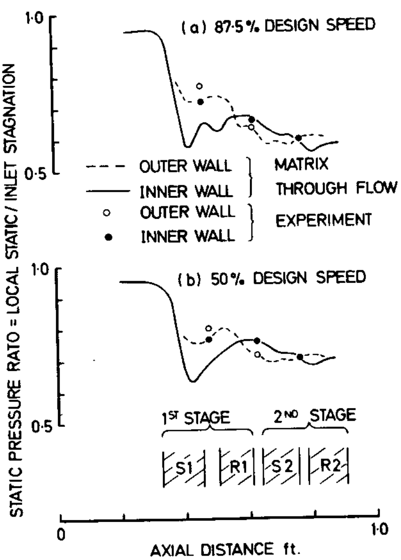


FIGURE 18.—*Static pressure for two-stage turbine.*

on annulus wall boundary layers shows that it should be possible to improve the predictions in the region of the end walls.

The turbine was fitted with static pressure tappings in the annulus walls. Figure 18 shows comparisons of observed pressure distributions with the theoretical calculations. The static pressure ratio is defined as the ratio of local static pressure to turbine inlet stagnation pressure. The main point to note is the presence of an inverse pressure gradient in the region of the second-stage stator blade—static pressure on the inner wall greater than that at the outer wall—which was successfully reproduced by the through-flow analysis. An alternative to the through-flow method is what is known as the streamline curvature duct flow method (ref. 5). Frost (ref. 24) has found that this method, which is widely used throughout the aircraft industry, did not predict the inverse pressure gradient. This example serves to show that when calculating the detailed internal aerodynamics, the flow inside the blade rows must be analyzed if a fairly accurate solution of the flow pattern is required.

Single-Stage Axial Flow Turbine

This single-stage, lightly loaded turbine was designed and tested at NGTE (ref. 25). In the initial through-flow analysis, no allowance was made for annulus wall boundary layers and the local polytropic efficiencies were assumed to be constant throughout the flow field. The predicted velocities (fig. 19) at turbine exit were in fair agreement with the observed values. Some measure of improvement in the region of the outer annulus wall was obtained by Herbert et al. (ref. 26) by allowing for the blockage caused by the boundary layers on the annulus walls. Improved matching of the experimental and predicted velocity profile would require a detailed boundary-layer analysis along the lines suggested by Gregory-Smith.

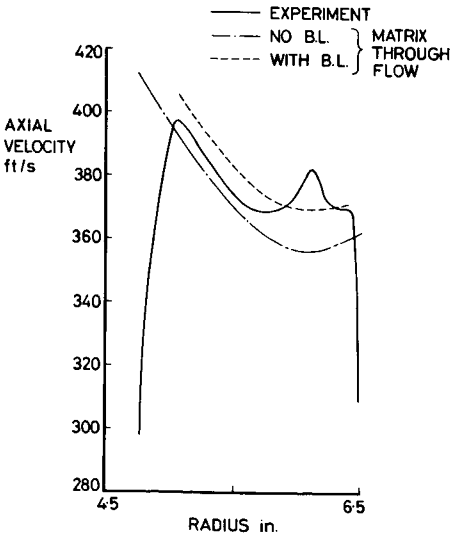


FIGURE 19.—*Axial velocity profile far downstream of single-stage turbine.*

Low Pressure Ratio Centrifugal Compressor

This centrifugal compressor was designed and tested by a firm in the United Kingdom. The results shown in figure 20 are the experimental and theoretical distributions of static pressure ratio along the shroud. The static pressure ratio is defined as the ratio of local to inlet static pressure. In performing the initial calculations, the values of local polytropic efficiency were assumed to be constant throughout the flow field and the slip factor equal to unity. The solution, although giving the correct trend, is in poor agreement with the observed pressures. By assuming a non-uniform distribution of local polytropic efficiency and a slip factor of 0.91, the matching between experiment and theory was improved. This example shows that if a scientifically based model for the flow loss can be formulated then the through-flow theory might eventually be used to provide a quantitative picture of the flow pattern.

High Pressure Ratio Centrifugal Compressor

This example of a centrifugal compressor has been included to illustrate the use of the through-flow program at the design stage of a machine. The initial and modified (final) hub-shroud profiles are shown in figure 21. The only difference between the two impellers is that for the modified machine, the inducer extends beyond the leading edge of the splitter vanes, thus giving a deeper inducer section. The relative Mach number distributions along the hub and shroud profiles are shown in figure 22. It will be seen that the severe velocity gradient in the region of the inducer

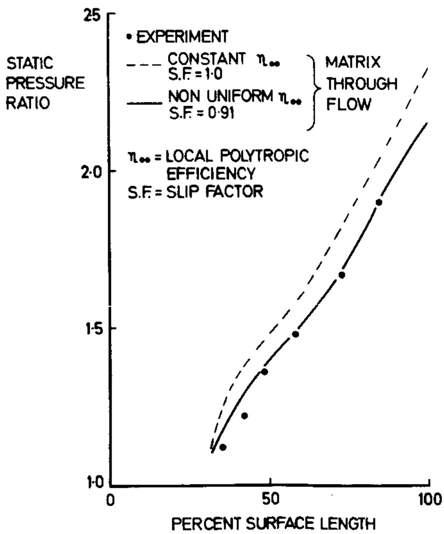
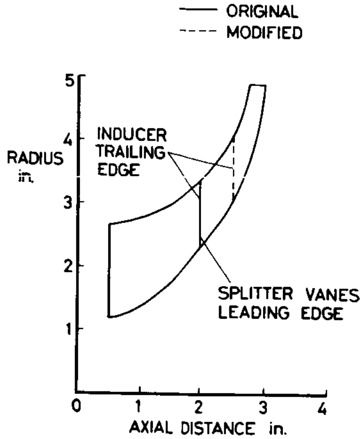


FIGURE 20.—*Static pressure distribution along the shroud of a low pressure ratio centrifugal compressor.*

FIGURE 21.—*Hub-shroud profile of high pressure ratio centrifugal compressor.*



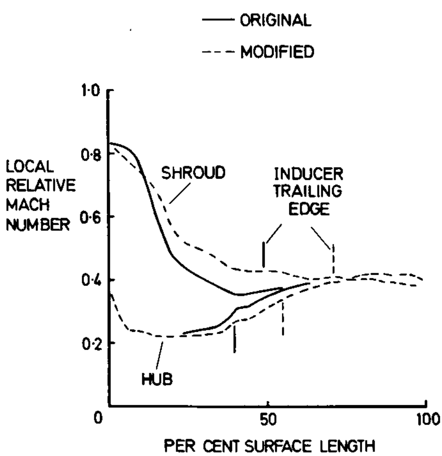


FIGURE 22.—*Hub-shroud Mach number distributions for high pressure ratio centrifugal compressor.*

leading edge for the original impeller is to some extent alleviated in the modified impeller. This example demonstrates the effects of modifications that are possible within the limits of the same inlet and outlet areas and overall length of the machine. A new design may, of course, permit variations on all these factors and the use of a computer method helps in choosing the best combination.

CONCLUSIONS

Computer solutions for the meridional and blade-to-blade flow patterns in turbomachines have been described. The theory is based on the earlier work of Wu (ref. 11) and the numerical solution is obtained by finite difference approximations to the governing equations. The main conclusions are the following.

Blade-to-Blade Flow

(1) Comparisons with exact cascade solutions show that the blade-to-blade program gives an accurate solution for incompressible flow.

(2) Analysis of turbine rotor blade sections shows that detailed flow patterns can be obtained which would probably be very difficult to find experimentally.

(3) A comparison with experimental data for a turbine stator blade shows that the method gives a good estimate of high subsonic flow. This analysis demonstrates that the asymmetric finite difference grid developed here is an advancement over the conventional square or rectangular grid.

(4) An example of a two-stage turbine illustrates that the three-dimensional pressure distributions can be predicted quite well.

Meridional Flow

(1) The matrix through-flow theory has enabled significant advances to be made in calculating meridional flow patterns. An analysis of a two-stage turbine shows that the theory gives a good estimate of annulus wall static pressure distributions.

(2) An example of a centrifugal compressor shows that small modifications to the impeller can have significant effects on the flow field. This analysis demonstrates that computer methods can help in selecting the "best" geometry.

(3) A simple calculation of annulus wall boundary layers for a single-stage turbine enables the through-flow predictions to be improved by allowing for the blockage caused by the boundary layers.

(4) Improved matching between experimental and predicted flow profiles depends on finding a better loss model and an accurate solution for the boundary-layer development along the annulus walls.

ACKNOWLEDGMENT

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LIST OF SYMBOLS

Q	Mass flow
r, z, ϕ	Radial, axial, and circumferential coordinates
V	Absolute velocity
W	Relative velocity
x, y	Coordinates with tilted axes
λ, α, μ	Flow angles
ψ	Stream function

Subscripts

d	Far downstream of blade row
u	Far upstream of blade row
x	x-component
y	y-component
ϕ	Circumferential component

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DISCUSSION

T. KATSANIS (NASA Lewis Research Center): Mr. Smith has shown that approximate three-dimensional solutions for flow through a turbomachine can be obtained by a combination of meridional and blade-to-blade solutions. This is fairly well known. However, we have here a fair number of examples showing both the strengths and weaknesses of these methods in applications.

Limitations of the method should be clearly stated. It appears that the flow must be absolutely irrotational, steady relative to the rotating blades, and nonviscous, and that the flow must be complete subsonic. There must be other assumptions. Certain advantages are stated for the method and the program, but the basic assumptions and limitations are not specified.

The author should specifically state how his method differs from those currently available. For example, a nonorthogonal mesh is used, but the corresponding finite difference equation is not given. Another example is equation (8), where a term has been added, but no explanation of its significance or why $q(x,\phi)$ is not also a function of $\bar{\partial}\psi/\partial x$.

Some comments must be made on one error. This is the statement that the boundary condition of zero velocity normal to the blade is not satisfied in references 8 and 9. This statement is not true. Further, the statement is made that finite difference stars with short limbs leads to large truncation errors. It is true that the standard finite difference equation for unequal spacing has a larger truncation error than with equal spacing. However, this does not mean that the error in the solution will be larger. In fact, with a rectangular mesh, there is theoretically no loss in the accuracy of the solution due to an irregular boundary. This has been amply demonstrated by extensive use and experimentation with the programs of references 8 and 9.

Mr. Smith does not discuss the reason for the use of iterative or relaxation methods for solving matrix equations. There are two main reasons, one being the numerical stability which can be controlled by using a suitable rigorously calculated overrelaxation factor, which assures numerical stability with an optimum rate of convergence. The other reason is economy of storage, which is not shared by most direct methods. Certainly numerical stability cannot be improved with a direct method. The advantage of a direct method would be in reducing the computer calculation time. This reduction in computer time could conceivably be a real advance, provided that storage requirements are not significantly increased, and that, as indicated, numerical stability is achieved. I would be very much interested in seeing a comparison of computer times for Mr. Smith's program with the times for a method using optimum overrelaxation.

M. D. WOOD (Cambridge University): The paper indicates the magnitude of the recent advances made in calculating fluid flow in turbomachines. The examples given show that compressible flow in turbine configurations can be predicted to a high degree of accuracy. However, as soon as compressor-type machinery is considered, the position is not so satisfactory. No one is really surprised, because the influence of boundary-layer growth and separation in compressor flow is likely to introduce effects which are of dominant importance. These effects are only recognized in the Wu equations through the presence of losses, and in general even these losses represent average or "smeared" values taken over appropriate computing planes.

It is clear that there are few shortcomings in the equations of motion which Wu manipulates—the shortcomings are only in the simplifications we impose in order to obtain quick gains in current predictive accuracy. I therefore suggest that we should now have the courage to involve ourselves in combining the currently developing detailed calculations of the viscous effects in turbomachinery with the type of basic Wu program described by Mr. Smith. To take examples, we can see how boundary-layer separation in blade corners will lead to warping of stream surfaces. This warping can, in principle, be incorporated in the Smith-type programs. Again, incorporating the predicted development of the boundary layer on blade surfaces would give better understanding of the "slip" factor for use in investigations of centrifugal compressors. Finally, inclusion of the turbulent diffusion effects between adjacent fluid layers would lead to more realistic representation of the fluid forces in the Wu-type equations.

Although this sounds like a daunting program, it is no more daunting than the thought, 10 years ago, of putting Wu's equations on a computer. Perhaps the author would put my hopes into perspective by explaining what he intends to do next.

A. S. MUJUMDAR (Carrier Corporation): As pointed out by Dr. Katsanis, since the program itself is apparently the major contribution made by the author, it is unfortunate that it cannot be released for publication. Any comparison with the generally available Katsanis programs must, therefore, remain one-sided. The overrelaxation procedure using proper grid spacing and an optimized overrelaxation factor should yield numerically stable results for well-guided geometries. As suggested by Wilkinson (ref. D–1), the maximum relative velocity change between successive iterations should be taken as the criterion for convergence rather than the maximum streamline deviation chosen by Katsanis.

Since reference 12 in the author's paper is not readily accessible, may I suggest that the finite difference analysis and the numerical scheme be included as an appendix to the paper when it is published. To my knowledge there is no "conventional" finite difference scheme to solve the Poisson-type partial differential equations; a number of variations are possible.

Referring to figure 15 of the paper, could the author explain why the two-dimensional matrix solution appears to give better agreement than the three-dimensional solution with the experimental data for the pressure surface.

Finally, I wish to bring to the author's attention the experimental study of the flow in the blade passages of a radial turbine reported by Glenny (ref. D–2), which may be used to provide further checks for the computer code.

R. C. DEAN (Creare Inc.): I'm a little bit disturbed by perhaps the implications that you suggest about the use of potential analysis in centrifugal compressors. In my experience, the potential analysis usually considerably overpredicts the pressure rise in the wheel and underpredicts or predicts a low relative Mach number at the discharge of the impeller. We have found this through several comparisons between these solutions and data. The potential analysis you are suggesting implies, I think, that the flow follows the blading. I think it is very misleading to think that such a solution would work toward the back of the impeller. The important physics of the flow are not included in the analysis.

L. MEYERHOFF (Eastern Research Group): I have a number of questions.

(1) Do you have any convergence criteria?

(2) Are you able to predict the number of iterations for the convergence criteria?

(3) How do you determine the trailing edge flow angle?

(4) Did the addition of the E term referred to in your paper still keep the equations set up by Wu, exact? It is not clear whether the equations are still exact after you add these E terms.

(5) Do you know of any analytical proof of the truncation error for the overrelaxation referred to in your paper?

SMITH (Author): The author thanks the five discussors for their review of this paper.

First, taking the specific points raised by Dr. Katsanis, the purpose of this paper was to present numerical solutions for several turbomachines to indicate the magnitude of the recent advances made in calculating the fluid mechanics rather than a detailed account of the mathematics of the flow models. The limitations of the methods and the finite difference approximations have been published in references 4 and 10.

On the question of boundary conditions, I have perhaps not made my point clear. In Dr. Katsanis' method (refs. 8 and 9) the flow domain is covered with a rectangular or square grid (fig. D–1). To obtain the blade surface velocity at points such as A, the circumferential component of velocity, W_ϕ, is obtained from the relationship

$$W_\phi = -\frac{1}{b\rho}\frac{\partial\psi}{\partial m}$$

and the resultant surface velocity W is determined so that there is zero velocity normal to the blade; thus,

$$W = \frac{W_\phi}{\cos\beta} \tag{D-1}$$

where β is the local blade surface angle. For points such as B the meridional component of velocity, W_m, is obtained from the relationship

$$W_m = \frac{1}{b\rho r}\frac{\partial\psi}{\partial\phi}$$

and the resultant velocity is given by

$$W = \frac{W_m}{\sin\beta} \tag{D-2}$$

I agree with Dr. Katsanis that these relationships ensure zero velocity normal to the blade. However, a restriction is placed on the resultant velocities; equation (D–1) is limited to $|\beta| \leq 60°$ and equation (D–2) is limited to $|\beta| \geq 30°$. This, I feel, implies an error in the derivatives of the stream function or the stream function values and when the condition of zero normal is imposed gives rise to an error in the resultant velocity.

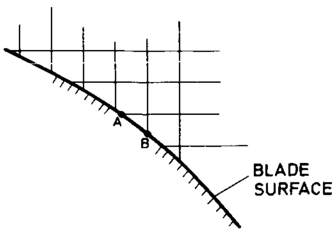

FIGURE D–1.—*Square finite difference grid.*

Turning to the problem of the solution of the matrix equations, I must admit that I have no experience of iterative or relaxation methods. The method I have adopted is a direct method developed by the National Physical Laboratory (NPL), England. This approach, which removes one possible source of divergence, has proved to be very stable numerically and, as I pointed out in my paper, only the band of nonzero elements are formed and stored in the computer. In the recent blade-to-blade computer program, the storage requirements have been further reduced by making use of magnetic tapes as backing store; if the width of the band matrix is W, the core store required is approximately $8W^2$. There is a further point concerned with the basic techniques of the method of solution. Unlike "conventional" direct methods, the NPL procedures do not "invert" the matrix on every iteration; the solution is obtained by a backward and forward substitution process. The matrix equation to be solved is

$$[M]\cdot[\psi]=[q]$$

The first step is a decomposition of the matrix $[M]$; thus

$$[L]\cdot[U]\cdot[\psi]=[q]$$

where the matrices $[L]$ and $[U]$ are lower and upper triangular band matrices, which are only computed on the first iteration. The solution is then obtained by (1) a process of forward substitution, solving for $[Z]$ from

$$[L]\cdot[Z]=[q]$$

and (2) a process of backward substitution, solving for $[\psi]$ from

$$[U]\cdot[\psi]=[Z]$$

The direct method provides an exact solution for the matrix equations and so it could be argued that, since the overall process for finding the stream function distribution is an iterative procedure, it is not necessary to obtain an exact solution on the earlier iterations. It may well be that the best approach is a relaxation method on the earlier iterations, making no attempt to reduce the residuals to zero on each iteration, followed by a direct method on the final iterations.

In answer to Dr. Wood, I would agree that we should now have the courage to extend the type of calculations I have described to include viscous effects. However, as is inevitable with an advanced calculation procedure, my experience of the use of the matrix methods has shown that, for the computer programs to become basic design tools, effort is also required in generating supporting programs for preparing geometric input data and graphical display of output data. This is one aspect I intend to examine.

I am grateful to Mr. Mujumdar for drawing my attention to experimental investigation of a radial turbine. With regard to the solutions for the turbine stator (fig. 15) I am unable to provide an explanation for the two-dimensional solution being in better agreement with the pressure surface experimental data than the three-dimensional solution.

I agree with Mr. Dean that, in the case of centrifugal compressors, some of the important physics of the flow are not included in the analysis. However, I feel that, even with the assumption that the flow follows the blading, the computer tools can help the designer in selecting the best geometry. If a boundary-layer analysis of a potential flow velocity distribution indicates, for example, separation in the inducer of a centrifugal compressor then I am sure Mr. Dean would agree with me that the designer would modify the geometry to overcome this problem. A number of examples illustrating this point are given by Dallenbach (ref. D–3) and Ball et al. (ref. D–4).

In answer to Mr. Meyerhoff, it is difficult to establish a unique convergence criterion. The criteria I have adopted are

$$\mathrm{TOL}=\frac{\psi-\psi_{p-1}}{\psi_{p-1}} \qquad \text{(through-flow method)}$$

$$\mathrm{TOL}=\frac{\psi-\psi_{p-1}}{Q} \qquad \text{(blade-to-blade method)}$$

It has been found that TOL can be reduced to 0.001 in 15 iterations for the through-flow method and 0.0001 in 10 to 30 iterations for the blade-to-blade method.

The calculation of the flow angle is a problem I have avoided by assuming it can be determined from existing empirical rules for the deviation. Clearly this is unsatisfactory, as I have indicated by the turbine example of figure 10. In a real flow the circulation is determined by viscous effects, particularly for compressor-type machinery as illustrated by figure 16. This is one aspect of turbomachinery fluid mechanics that demands research.

The addition of the $E(\bar{\partial}\psi/\partial x)$ does not change the basic Wu equations. This term was added to keep the width of the band matrix to a minimum.

Finally, on the question of the truncation error for conventional finite difference analysis, which Dr. Katsanis also raised, I can perhaps best illustrate my point by considering a square grid. For such a grid a five point star (refs. 8 and 9) is adopted to represent the Laplacian operator. Consider a star near to a boundary (fig. D–2) with one irregular limb. It will be supposed that for the star center, point 3,

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_3+\left(\frac{\partial^2 f}{\partial y^2}\right)_3=\sum_{n=1}^{S} a_n f_n+T \qquad \text{(D–3)}$$

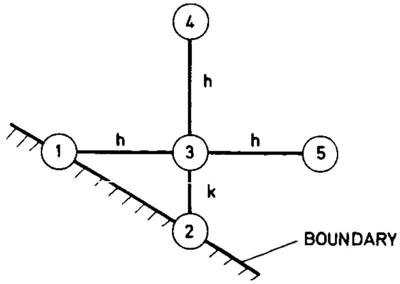

FIGURE D-2.—*Five-point star with one irregular limb.*

where T is a truncation term and a_1, a_2 are the weightings. The function f can be expanded as a two-dimensional Taylor series about the point 3 at which the function has the value f_3 and derivatives $f_{3,x}, f_{3,y}, f_{3,xx}$, etc. Substituting the Taylor series expressions into equation (D-3), it follows that

$$
\begin{aligned}
f_{3,xx}+f_{3,yy}=a_1&\left(f_3-hf_{3,x}+\frac{h^2}{2}f_{3,xx}-\frac{h^3}{6}f_{3,xxx}+\ldots\right)\\
&+a_2\left(f_3-hf_{3,y}+\frac{h^2}{2}f_{3,yy}-\frac{h^3}{6}f_{3,yyy}+\ldots\right)\\
&+a_3 f_3\\
&+a_4\left(f_3+kf_{3,y}+\frac{k^2}{2}f_{3,yy}+\frac{k^3}{6}f_{3,yyy}+\ldots\right)\\
&+a_5\left(f_3+hf_{3,x}+\frac{h^2}{2}f_{3,xx}+\frac{h^3}{6}f_{3,xxx}+\ldots\right)\\
&+T'
\end{aligned}
$$

Since f is a general function, it follows that the coefficients of f and each of its derivatives may be equated on each side of the above equation. There are five disposable constants $a_i(i=1(1)5)$ and so only the coefficients of f, f_x, f_y, f_{xx} and f_{yy} may be used in order to make the finite difference approximation independent of the low-order derivatives. From the coefficient equations, it is easy to show that the solution for the weightings is

$$a_1=\frac{1}{h^2}$$

$$a_2=\frac{2}{(\gamma+1)h^2}$$

$$a_3=\frac{2(\gamma+1)}{\gamma h^2}$$

$$a_4=\frac{2}{(\gamma+1)\gamma h^2}$$

$$a_5=\frac{1}{h^2}$$

where $\gamma=k/h$.

The truncation error is

$$T'=\frac{h(\gamma^2-1)}{3(\gamma+1)}f_{3,yyy}+0(h^2f_{3,yyyy}) \qquad \text{(D–4)}$$

It is seen, therefore, that as the irregular limb gets shorter (i.e., γ decreases) the truncation error increases.

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